Polymorphic Types (in System F — Girard – Reynolds)

$\vdash \lambda x : \mathbb{N} . x : \mathbb{N} \to \mathbb{N}$

$\vdash \lambda x : \mathbb{N} \times \mathbb{N} . x : (\mathbb{N} \times \mathbb{N}) \to (\mathbb{N} \times \mathbb{N})$

$\circ \quad \circ \quad \circ$

$\vdash \lambda x : \tau . x : \tau \to \tau$

for any type τ

$\vdash \lambda x : \tau \times \sigma . \pi_1(x) : \tau \times \sigma \to \tau$

for any types τ, σ

$\vdash \lambda x : (\tau \to \sigma) \times \tau . \pi_1(x)(\pi_2(x))$

$: (\tau \to \sigma) \times \tau \to \sigma$

for any types τ, σ

Let $TV = \{\alpha, \beta, \ldots\}$ a set of type variables

These replace the basic types: $BT = TV$

New constructs in $\lambda^{\forall}$:

type: $\forall \alpha . \tau$ (also: $\prod_{\alpha : *} \tau$)

term (intro): $\Lambda \alpha . t$ (also: $\lambda \alpha : * . t$)

term (elim): $t[\tau]$ (also: $(t\,\tau)$)

Typing rules:

α not free inside Γ

$$\frac{\Gamma \vdash t : \tau \qquad \alpha \notin fv(\Gamma)}{\Gamma \vdash \Lambda\alpha.t : \forall\alpha.\tau}\ (\forall I)$$

$$\frac{\Gamma \vdash t : \forall\alpha.\tau}{\Gamma \vdash t[\sigma] : \tau\{\sigma/\alpha\}}\ (\forall E)$$

Computational rules:

- α-conversion of $\forall\alpha.\tau$
  (e.g. $(\forall\alpha.\ \alpha \times \mathbb{N}) \simeq_\alpha (\forall\beta.\ \beta \times \mathbb{N})$

- $(\Lambda\alpha.t)[\tau] \longrightarrow_\beta t\{\tau/\alpha\}$
- $(\Lambda\alpha.t[\alpha]) \longrightarrow_\eta t$ if $\alpha \notin fv(t)$

$\vdash \Lambda\alpha.\ \lambda x:\alpha.\ x \quad : \forall\alpha.\ \alpha \to \alpha$

$\vdash \Lambda\alpha.\ \lambda x:\alpha.\ \lambda y:\alpha.\ x$ ($\leftarrow (\circ\ y)$)
$\quad : \forall\alpha.\ \alpha \to \alpha \to \alpha$

$\vdash \Lambda\alpha.\ \Lambda\beta.\ \lambda x:\alpha.\ \lambda y:\beta.\ x$
$\quad : \forall\alpha.\ \forall\beta.\ \alpha \to \beta \to \alpha$

$\vdash \Lambda\alpha.\ \Lambda\beta.\ \Lambda\gamma.$
$\lambda f:\beta \to \gamma.\ \lambda g:\alpha \to \beta.$
$\lambda x:\alpha.\ f(g\,x)$
$: \forall\alpha,\beta,\gamma.\ (\beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$

(With $\mathbb{N}$ and $\times$)
$\vdash \lambda f:(\forall\alpha.\ \alpha \to \alpha).$
$\langle f[\mathbb{N}](5),\ f[\mathbb{N}\times\mathbb{N}](\langle 7,4\rangle)\rangle$
$: (\forall\alpha.\ \alpha \to \alpha) \to (\mathbb{N} \times (\mathbb{N} \times \mathbb{N}))$

Standard results:
- Church - Rosser
- Strong normalization
- Subject Reduction
- Type Uniqueness

$\lambda^{\forall, \times, +, 1, 0, \to}$ can be encoded in $\lambda^{\forall, \to}$

Church encodings for $\times, +, 0, 1$
(Warning: do NOT memorize this, we will see a more general approach later on).

Encoding 1

$1 \longmapsto \forall \alpha. \alpha \to \alpha$

$\bullet \longmapsto \Lambda \alpha. \lambda x : \alpha. x$

Encoding 0

$0 \longmapsto \forall \alpha. \alpha$

$\text{absurd}_\tau(e) \longmapsto e[\tau]$

Encoding $\tau \times \sigma$

$\tau \times \sigma \longmapsto \forall \alpha. (\tau \to \sigma \to \alpha) \to \alpha$

$\langle t, e \rangle \mapsto \Lambda \alpha. \lambda f : \tau \to \sigma \to \alpha. f\ t\ e$
(with $t : \tau$, $e : \sigma$)

$\pi_1(p) \mapsto p[\tau](\lambda x : \tau. \lambda y : \sigma. x)$

-- 2 ----- σ ---------- y

$\ulcorner \pi_1(\langle t, e \rangle) \urcorner =$

$(\Lambda\alpha.\, \lambda f : (\tau \to \sigma \to \alpha).\, f\, t\, e)[\tau]$

$(\lambda x : \tau.\, \lambda y : \sigma.\, x) =$

$(\lambda f : (\tau \to \sigma \to \tau).\, f\, t\, e)(\lambda x : \tau.\, \lambda y : \sigma.\, x) =$

$(\lambda x : \tau.\, \lambda y : \sigma.\, x)\, t\, e =$

$(\lambda y : \sigma.\, t)\, e = t$

$\Rightarrow$ β law is rejected

Ex: compare this encoding to the one for untyped λ

Encoding $+$

$\tau + \sigma \mapsto \forall\alpha.\, (\tau \to \alpha) \to (\sigma \to \alpha) \to \alpha$

$in_1(t) \mapsto \Lambda\alpha.\, \lambda f : \tau \to \alpha.\, \lambda g : \sigma \to \alpha.\, f\, t$

$in_2(e) \mapsto \Lambda\alpha.\, \lambda f : \tau \to \alpha.\, \lambda g : \sigma \to \alpha.\, g\, e$

(where $t : \tau$, $e : \sigma$)

case h of $in_1(x) \to t$; $in_2(y) \to e$

$\mapsto$

$h[\gamma]\, (\lambda x : \tau.\, t)\, (\lambda y : \sigma.\, e)$

(where (case ...) : γ)

Ex: verify the β laws for $+$.