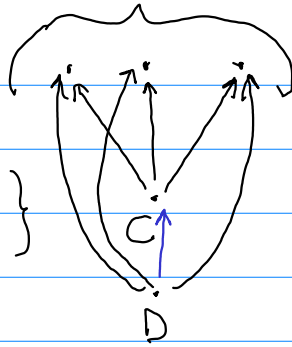


$$\mathcal{X} \subseteq |\mathcal{C}|$$

cone $\mathcal{X}$ (simplified)

category:

$$|\text{cone } \mathcal{X}| = \left\{ (C, \xi_-) \mid \begin{array}{l} \text{cone} \\ \text{on } \mathcal{X} \end{array} \right\}$$



$$\text{cone } \mathcal{X}((D, g_-), (C, \xi_-))$$

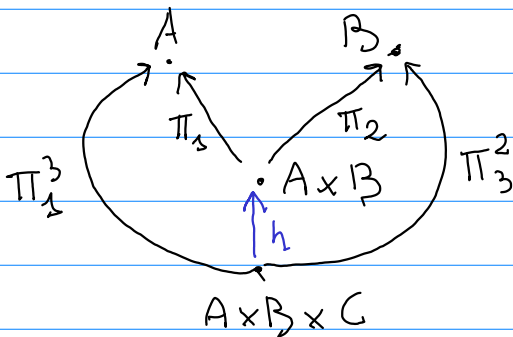
$$= \{ h: D \rightarrow_C C \mid \forall X \in \mathcal{X}. g_X = h; \xi_X \}$$

upper bound $\text{ub}(X)$ in $(A, \leq)$, $X \subseteq A$
preorder

$\equiv$ lower bound in $(A, \leq)^{\text{op}}$

co-cone in $\mathcal{C} \equiv$ cone in $\mathcal{C}^{\text{op}}$

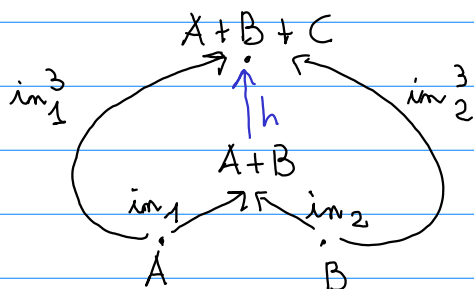
Ex: in Set $\mathcal{X} = \{A, B\}$



$$h(a, b, c) = (a, b)$$

Let $A+B$ be the disjoint union

$$A+B = (\{1\} \times A) \cup (\{2\} \times B)$$



$$\text{im}_1(a) = \langle 1, a \rangle$$

$$\text{im}_2(b) = \langle 2, b \rangle$$

$$h(\langle 1, a \rangle) = \langle 1, a \rangle$$

$$h(\langle 2, b \rangle) = \langle 2, b \rangle \quad \left(\begin{array}{c} \text{no} \\ \langle 3, c \rangle \end{array} \right)$$

$(A, \subseteq)$ preorder

x is a "bottom" element $\equiv \forall a \in A. x \subseteq a$

x "top" $\equiv x$ bottom in $(A, \subseteq)^{\text{op}}$

$\perp$ bottom, $\top$ top

lem: bottoms are unique, up to $\sim$

where $x \sim y \equiv x \subseteq y \subseteq x$

Def: a preorder $(A, \subseteq)$ is a poset

iff $x \sim y \Rightarrow x = y$

(antisymmetry)

lem: a preorder up to $\sim$ is a poset

poset $\rightsquigarrow$ cat up to iso

cat th: $\perp \rightsquigarrow$ initial object

$X \in |\mathcal{C}|$ is weakly initial

$$\equiv \forall C \in |\mathcal{C}|. \exists f: X \rightarrow C$$

$X \in |\mathcal{C}|$ is initial

$$\equiv \forall C \in |\mathcal{C}|. \exists ! f: X \rightarrow C$$

final / terminal in $\mathcal{C} \equiv$ initial in $\mathcal{C}^{op}$

Ex: In Set $\emptyset$ initial, $\forall a. \{a\}$ final

In Top " " "

In Vect $_{\mathbb{K}}$ $\{0\}$ initial and final

In Grp $\{e\}$ " " "

Notation: 0 initial

1 final

$$!_A: 0 \rightarrow A \quad !^A: A \rightarrow 1$$

lem A, B initial $\Rightarrow A \cong B$ (iso)

$$\begin{array}{ccc} !_A^A \hookrightarrow A & \xrightarrow{!_B^A} & B \\ & \xleftarrow{!_A^B} & \\ & !_A^B & \end{array}$$

$$!_B^A, !_A^B = !_A^A = \text{id}_A \quad (\& \text{ symm.})$$

$\min^A(X) \equiv \text{the } \perp \text{ in } (X, \subseteq \cap X^2)$

$\max \equiv \min \text{ in } (A, \subseteq)^{\text{op}}$

($\min, \max$ may not exist)

$\inf^A(X) = \prod^A X \equiv \max^A(\text{lb}^A(X))$

$\sup^A(X) = \sqcup^A X = \prod X \text{ in } (A, \subseteq)^{\text{op}}$

Cat th: $\inf \rightsquigarrow$ final cone

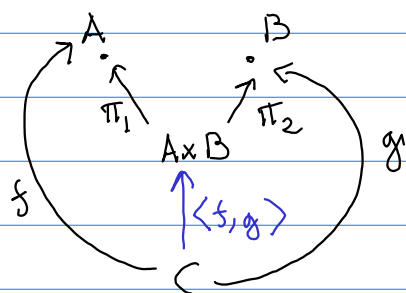
$\sup \rightsquigarrow$ initial cocone

In our simplified setting

$\inf \rightsquigarrow$ product

$\sup \rightsquigarrow$ coproduct (or sum)

In Set $A \times B$ is the product

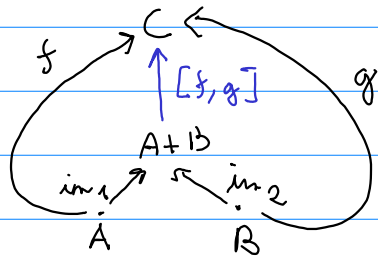


$$\langle f, g \rangle(c) = \langle f(c), g(c) \rangle$$

is the unique morphism s.t.

$$\langle f, g \rangle; \pi_1 = f \quad \langle f, g \rangle; \pi_2 = g$$

In Set $A+B$ is the coproduct



$$[f, g](\langle 1, a \rangle) = f(a)$$

$$[f, g](\langle 2, b \rangle) = g(b)$$

is the unique morphism s.t.

$$in_1; [f, g] = f \quad in_2; [f, g] = g$$

In Top $A \times B$ with the product topology
is the product (as in cat. th.)

In Grp $A \times B$ (direct product)
is the product (as in cat. th.)

In Vect $_K$...

In Vect $_K$ $A + B \cong A \times B$

In Grp $A + B \cong A * B$
free product
 $a b a' b' \dots a'' b''$

$\mathcal{X}$ infinite

is Set $\prod \mathcal{X}$, $\coprod \mathcal{X}$ (disj. union)

in Vect $_K$ $\prod \mathcal{X}$ is the product

$$\vec{x} = \langle x_1, x_2, \dots \rangle$$

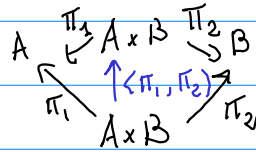
$\coprod \mathcal{X}$ is $\subseteq \prod \mathcal{X}$

$\vec{x}$ with finite support

2 with $A \times B$

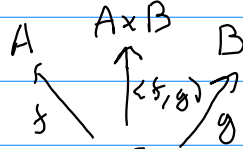
$$1) \langle \pi_1, \pi_2 \rangle = \text{id}_{A \times B}$$

↑ is a cone morph.



$$\langle \pi_1, \pi_2 \rangle; \pi_1 = \pi_1$$

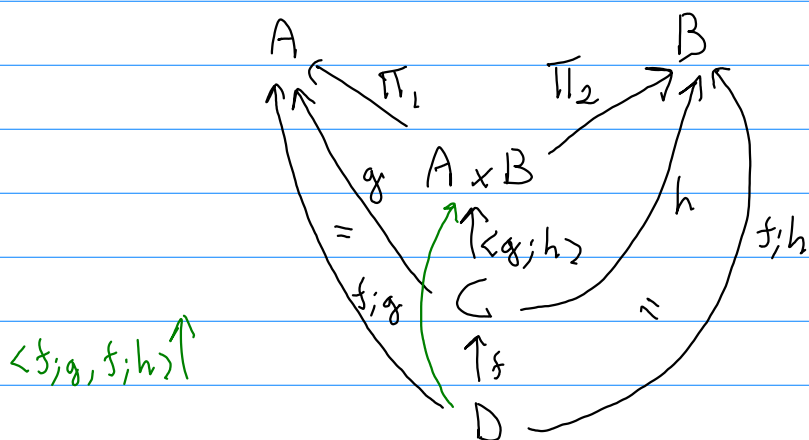
(& for π_2)



but $\exists!$ cone mor. $A \times B \rightarrow A \times B$

by finality, which is $\text{id}_{A \times B}$

$$2) f; \langle g, h \rangle = \langle f; g, f; h \rangle$$



We verify that $f; \langle g, h \rangle$ is a core morphism $D \rightarrow A \times B$, then by finality $f; \langle g, h \rangle = !^D = \langle f; g, f; h \rangle$

$$\underbrace{f; \langle g, h \rangle; \pi_1}_{= g} \stackrel{?}{=} f; g \quad (\& \text{ for } \pi_2)$$

$$f, g: C \rightarrow A \times B$$

$$3) \begin{cases} f; \pi_1 = g; \pi_1 \\ f; \pi_2 = g; \pi_2 \end{cases} \Rightarrow f = g$$

$$f = f; \text{id}_{A \times B} \stackrel{\textcircled{1}}{=} f; \langle \pi_1, \pi_2 \rangle$$

$$\stackrel{\textcircled{2}}{=} \langle f; \pi_1, f; \pi_2 \rangle \stackrel{\text{HP}}{=} \langle g; \pi_1, g; \pi_2 \rangle$$

$$\stackrel{\textcircled{2}}{=} g; \langle \pi_1, \pi_2 \rangle \stackrel{\textcircled{1}}{=} g; \text{id}_{A \times B} = g$$